

Toric data and Killing forms on homogeneous Sasaki-Einstein manifold $T^{1,1}$

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March 3, 2015

Abstract

Throughout this paper we investigate the complex structure of the conifold $C(T^{1,1})$ basically making use of the interplay between symplectic and complex approaches of the Kähler toric manifolds. The description of the Calabi-Yau manifold $C(T^{1,1})$ using toric data allows us to write explicitly the complex coordinates and apply standard methods for extracting special Killing forms on the base manifold. As an outcome, we obtain the complete set of special Killing forms on the five-dimensional Sasaki-Einstein space $T^{1,1}$.

1 Introduction

Symmetries are widely used as a useful tool in modeling physical systems. The ordinary symmetries are associated with isometries, that are spacetime diffeo-

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morphisms that leave the metric invariant. A one-parameter continuous isometry is connected with a Killing vector field. An extension of the Killing vector fields is represented by conformal Killing vector fields [23] with flows preserving a given class of metrics.

However, it has been proved that the investigation of symmetries in the whole phase space of a system is exceedingly useful. Such transformations of the whole phase space for which the dynamics of the system is left invariant are often referred as *hidden symmetries*. The hidden symmetries of curved manifolds are represented by Killing tensors and Killing-Yano tensors. Thanks to such symmetries many complicated physical problems become tractable taking into account that the equation of motion are separable and integrable. Analogously, conformal Killing-Yano tensors are associated with conserved quantities along null geodesics and integrability of massless field equations.

The purpose of this paper is to present a method to construct Killing forms on toric Sasaki-Einstein manifolds. We exemplify the procedure in the case of the five-dimensional homogeneous Sasaki-Einstein manifold $T^{1,1}$.

Until recently the only explicitly known non-trivial Sasaki-Einstein metric in dimension five was $T^{1,1}$ [5]. The five-dimensional manifolds $T^{p,q}$ which are the coset spaces $(SU(2) \times SU(2))/U(1)$ have been considered by Romans [16] in the context of Kaluza-Klein supergravity. Romans found that for $p = q = 1$ the compactification preserves 8 supersymmetries, while for other p and q all supersymmetries are broken.

In light of the *AdS/CFT* correspondence the $AdS_5 \times T^{1,1}$ model of [10] is the first example of a supersymmetric holographic theory based on a compact manifold which is not locally S^5 .

The approach we take in order to achieve our goal basically relies on the interplay between symplectic and complex coordinates on toric manifolds [13, 1]. In our particular case, this description of the conifold is slightly different from [11, 8], when the geometric features of the conifold are mainly exhibited. In turn, our approach enable us to use the correspondence between special Killing forms and parallel forms on the metric cone which was introduced by Semmelmann [18]. For more details concerning this method in the case of toric manifolds we refer to [22, 20].

The paper is organized in the following manner: In the second Section we introduces the main concepts and technical tools we use in the rest of the paper. We also give the necessary preliminaries regarding the special Killing forms, toric Sasaki-Einstein manifolds and the way their Calabi-Yau cones spaces can be regarded as complex manifolds. In Section 3 we apply these results constructing complex coordinates in the particular case of the conifold $C(T^{1,1})$. In Section 4 we extract the complete set of special Killing forms on the Sasaki-Einstein space $T^{1,1}$. Finally, our conclusions are presented within the last Section.

2 Preliminaries

2.1 Special Killing forms

A natural generalization of conformal Killing vector fields is given by the conformal Killing forms which are sometimes referred as twistor forms or conformal Killing-Yano tensors.

Definition 1. *A conformal Killing-Yano tensor of rank p on a n -dimensional Riemannian manifold (M, g) is a p -form ψ which satisfies*

$$\nabla_X \psi = \frac{1}{p+1} X \lrcorner d\psi - \frac{1}{n-p+1} X^* \wedge d^* \psi, \quad (1)$$

for any vector field X on M .

Here we used the standard conventions: ∇ is the Levi-Civita connection with respect to the metric g , X^* is the 1-form dual to the vector field X , \lrcorner is the operator dual to the wedge product and d^* is the adjoint of the exterior derivative d .

In component notation, the conformal Killing-Yano tensor equation is given by

$$\nabla_{(i_1} \psi_{i_2) i_3 \dots i_{p+1}} = \frac{1}{n-p+1} \left(g_{i_1 i_2} \nabla_j \psi^j_{i_3 \dots i_{p+1}} - (p-1) g_{[i_3 (i_1} \nabla_j \psi^j_{i_2) i_4 \dots i_{p+1}]} \right).$$

We used round brackets to denote symmetrization over the indices within. For $p=1$ we recover the usual definition of a Killing vector:

$$\nabla_{(j} \psi_{i)} = 0.$$

If ψ is co-closed in (1), then we obtain the definition of a Killing-Yano tensor [23] which, in component notation, satisfies the equation:

$$\nabla_{(j} \psi_{i_1) i_2 \dots i_p} = 0.$$

A particular class of Killing forms is represented by the special Killing forms:

Definition 2. *A Killing form ψ is said to be a special Killing form if it satisfies for some constant c the additional equation*

$$\nabla_X (d\psi) = c X^* \wedge \psi,$$

for any vector field X on M .

It is worth mentioning the fact that the most known Killing forms are actually special.

There is also a symmetric generalization of the Killing vectors:

Definition 3. *A symmetric tensor $K_{i_1 \dots i_r}$ of rank $r > 1$ satisfying the generalized Killing equation*

$$\nabla_{(j} K_{i_1 \dots i_r)} = 0,$$

is called a Stäckel-Killing tensor.

The analogue of the conserved quantities associated with Killing vectors is given by the following proposition:

Proposition 1. *For any geodesic γ with tangent vector $\dot{\gamma}^i$*

$$Q_K = K_{i_1 \dots i_r} \dot{\gamma}^{i_1} \dots \dot{\gamma}^{i_r},$$

is constant along γ .

Let us note that there is an important connection between these two generalizations of the Killing vectors. To wit, given two Killing-Yano tensors ψ^{i_1, \dots, i_k} and σ^{i_1, \dots, i_k} there is a Stäckel-Killing tensor of rank 2:

$$K_{ij}^{(\psi, \sigma)} = \psi_{ii_2 \dots i_k} \sigma_j^{i_2 \dots i_k} + \sigma_{ii_2 \dots i_k} \psi_j^{i_2 \dots i_k}.$$

This fact offers a method to generate higher order integrals of motion by identifying the complete set of Killing forms.

2.2 Sasaki-Einstein manifolds

Definition 4. *An almost contact structure on a smooth manifold M is a triple (φ, B, η) , where φ is a field of endomorphisms of the tangent spaces, B is a vector field and η is a 1-form on M satisfying (see [17])*

$$\varphi^2 = -I + \eta \otimes B, \quad \eta(B) = 1. \quad (2)$$

We remark that many authors also include in the above definition the conditions that $\varphi B = 0$ and $\eta \circ \varphi = 0$, although these are deducible from (2) (see [2]).

A Riemannian metric g on M is said to be *compatible* with the almost contact structure (φ, B, η) if and only if the relation

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

holds for all pairs of vector fields X, Y on M . In this case (φ, B, η, g) is called an *almost contact metric structure*. Moreover, if the Levi-Civita connection ∇ of the metric g satisfies

$$(\nabla_X \varphi)Y = g(X, Y)B - \eta(Y)X,$$

for all vector fields X, Y on M , then (φ, B, η, g) is said to be a *Sasakian structure* [2].

It is also important to note that Sasakian geometry is in fact the odd-dimensional counterpart of Kähler geometry, since a Sasakian structure may be reinterpreted and characterized in terms of the metric cone as follows. The metric cone of a Riemannian manifold (M, g) is the Riemannian manifold $C(M) = (0, \infty) \times M$ with the metric given by

$$\bar{g} = dr^2 + r^2 g,$$

where r is a coordinate on $(0, \infty)$. Then M is a Sasaki manifold if and only if its metric cone $C(M)$ is Kähler [3]. We note that the one form η extends to a one form on $C(M)$ by $\eta(X) = \frac{1}{2}\bar{g}(B, X)$ and M is identified with the subset $r = 1$ of $C(M)$. In particular, the cone $C(M)$ is equipped with an integrable complex structure J defined by

$$Jr\partial_r = B, \quad JY = \varphi Y - \eta(Y)r\partial_r, \quad Y \in TM,$$

and a Kähler 2-form ω given by

$$\omega = \frac{1}{2}d(r^2\eta) = \frac{1}{2}dd^c r^2,$$

where $d^c = \frac{i}{2}(\bar{\partial} - \partial)$, both J and ω being parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of \bar{g} . Moreover, M has odd dimension $2n + 1$, where $n + 1$ is the complex dimension of the Kähler cone. Conversely, given any algebraic Kähler orbifold, we can naturally associate to it a quasi-regular Sasakian manifold [3].

An *Einstein manifold* is a Riemannian manifold (M, g) satisfying the Einstein condition

$$Ric_g = \lambda g, \tag{3}$$

for some real constant λ , where Ric_g denotes the Ricci tensor of g . Einstein manifolds with $\lambda = 0$ are called *Ricci-flat manifolds*. A *Sasaki-Einstein manifold* is a Riemannian manifold (M, g) that is both Sasaki and Einstein. We note that in the case of Sasaki-Einstein manifolds one always has (3) with the Einstein constant $\lambda = 2n$. We also remark that Gauss equation relating the curvature of submanifolds to the second fundamental form shows that a Sasaki manifold M is Einstein if and only if the metric cone $C(M)$ is Kähler Ricci-flat. In particular the Kähler cone of an Sasaki-Einstein manifold has trivial canonical bundle [4, 21]. We note that one of the most familiar example of homogeneous Sasaki-Einstein five-manifold is the space $T^{1,1} = S^2 \times S^3$ endowed with the following metric [11, 5]

$$ds^2(T^{1,1}) = \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2.$$

A *toric Sasaki manifold* M is a Sasaki manifold whose Kähler cone $C(M)$ is a toric Kähler manifold [7]. In particular, a five-dimensional toric Sasaki-Einstein manifold is a Sasaki-Einstein manifold with three $U(1)$ isometries: $G = \mathbb{T}^3$. In this case one can construct canonical coordinates based on the symplectic geometry of the cone $C(M)$ and specify the Sasaki-Einstein structure in terms of toric data together with a single function G , a symplectic potential, on the three-dimensional image of the momentum map [15]. It is known that one of the simplest example of a toric non-orbifold singularity is the conifold $C(T^{1,1})$, i.e. the Calabi-Yau cone over $T^{1,1}$.

On the other hand, according to Semmelman [18], there is a correspondence between special Killing forms defined on the Sasaki-Einstein manifold M and the

parallel forms defined on the metric cone $C(M)$. More exactly, a p -dimensional differential form Ψ is a special Killing form on M if and only if the corresponding form

$$\Psi_{cone} := r^p dr \wedge \Psi + \frac{r^{p+1}}{p+1} d\Psi, \quad (4)$$

is parallel on $C(M)$.

In particular, on a five-dimensional Sasaki manifold M with the Reeb vector field B and 1-form $\eta := B^*$, there are the following two special Killing forms:

$$\Psi_1 = \eta \wedge d\eta, \quad \Psi_2 = \eta \wedge (d\eta)^2. \quad (5)$$

Besides these Killing forms, there are two closed conformal Killing forms, also called $*$ -Killing forms, given by

$$\Phi_1 = d\eta, \quad \Phi_2 = (d\eta)^2. \quad (6)$$

Moreover, in the case of the Calabi-Yau cone $C(M)$ it follows that we have two additional Killing forms on M connected with the additional parallel forms of the cone given by the holomorphic complex volume form Ω of $C(M)$ and its conjugate [18].

2.3 Symplectic and complex coordinates on toric manifolds

Let us consider a toric Sasaki-Einstein manifold. In the spirit of [1] we use in our further considerations the symplectic (action-angle) coordinates (y^i, Φ^i) ; here the angular coordinates Φ^i will generate the toric action. The y^i coordinates are obtained using the momentum map $\mu = \frac{1}{2}r^2\eta$, with the correspondence

$$y^i = \mu(\partial/\partial\Phi^i). \quad (7)$$

The Kähler form ω can be written in the simple manner [1, 13]

$$\omega = dy^i \wedge d\Phi^i.$$

In turn, the corresponding Kähler metric on the cone $C(M)$ is constructed using the *symplectic potential* G , which is a strictly convex function $G = G(y)$ of homogeneous degree -1 in y [13, 1]. We get

$$ds^2 = G_{ij} dy^i dy^j + G^{ij} d\Phi^i d\Phi^j,$$

where the metric coefficients are computed

$$G_{ij} = \frac{\partial^2 G}{\partial y^i \partial y^j},$$

with $(G^{ij}) = (G_{ij})^{-1}$.

The complex structure J can be described using the above symplectic coordinates metric coefficients, namely

$$J = \begin{pmatrix} 0 & -G^{ij} \\ G_{ij} & 0 \end{pmatrix}.$$

Now we return to the construction of the symplectic potential G .

A classical result of Delzant associates to any Delzant polytope $P \in \mathbb{R}^n$ a close connected symplectic manifold M , together with a Hamiltonian \mathbb{T}^n action on the manifold, showing that the polytope turns out to be the image of the momentum map, $P = \mu(M)$. We remind that a Delzant polytope is a convex polytope such that there are n edges meeting at each vertex, each edge meeting at the vertex is of form $1 + tu_i$, where $u_i \in \mathbb{Z}^n$, and $\{u_i\}$ can be chosen to form a basis in \mathbb{Z}^n .

A Delzant polytope can be described by the inequalities

$$l_A(y) := \langle y, v_A \rangle \geq 0, \text{ for } 1 \leq A \leq d,$$

where $\{v_A\}$ are inward pointing normal vectors to the facets of the polytope, d is the number of facets [1, 7].

Remark 1. *In the case of the Calabi-Yau cone we take $C(M)$ to be Gorenstein which is a necessary condition to admit a Ricci-flat Kähler metric and M to admit a Sasaki-Einstein metric. For affine toric varieties it is well-known that $C(M)$ being Gorenstein is equivalent to the existence of a basis for the torus \mathbb{T}^n for which*

$$v_i = (+1, w_i), \quad (8)$$

for each a, \dots, d and $w_a \in \mathbb{Z}^{n-1}$ [11, 13].

If B is the Reeb vector field, let us point out the following relation which links this geometric object to the metric coefficients

$$B_i = 2G_{ij}y^j.$$

Now let us also define the affine function $l_B := \langle B, \cdot \rangle$, and $l_\infty := \langle \sum_A v_A, \cdot \rangle$. Then, the symplectic potential G can be written in terms of the toric data [1, 7]

$$G = G^{can} + G^B + h, \quad (9)$$

where

$$\begin{aligned} G^{can} &= \frac{1}{2} \sum_A l_A(y) \log l_A(y), \\ G^B &= \frac{1}{2} l_B(y) \log l_B(y) - \frac{1}{2} l_\infty(y) \log l_\infty(y), \end{aligned} \quad (10)$$

and h is a homogeneous function of degree 1 in variables y . In the general case, as G needs to satisfy the Monge-Ampère equation, the function h is added.

For a complete determination of the symplectic potential G it is necessary to compute the Reeb vector B and the function h . There are two known different algebraic procedures to extract the components of the Reeb vector from the toric data. According to the AdS/CFT correspondence the volume of the Sasaki-Einstein space corresponds to the central charge of the dual conformal field theory. The first procedure is based on *the maximization of the central charge* (a -maximization) [9] used in connection with the computation of the Weyl anomaly in 4-dimensional field theory. The second one is known as *volume minimization* (or Z -minimization) [13].

The symplectic potential allow us to pass to the coordinate patch (x^i, Φ^i) obtained from complex coordinates $z^i := x^i + i\Phi^i$, with $i := \sqrt{-1}$; this is possible via the Legendre transform which relates the symplectic potential G and the Kähler potential F

$$F(x) = \left(y^i \frac{\partial G}{\partial y^i} - G \right) : (y = \partial F / \partial x) .$$

Consequently F and G are Legendre dual to each other

$$F(x) + G(y) = \sum_j \frac{\partial F}{\partial x^j} \frac{\partial G}{\partial y^j} ,$$

and

$$x^i = \frac{\partial G}{\partial y^i}, \quad y^i = \frac{\partial F}{\partial x^i} .$$

The metric structure is now written in the following manner

$$ds^2 = F_{ij} dx^i dx^j + F_{ij} d\Phi^i d\Phi^j ,$$

where the metric coefficients are again obtained using the Hessian of the Kähler potential F , i.e.

$$F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j} .$$

Note also that $(F_{ij}) = (G^{ij})$ [1].

With respect to the coordinates (x^i, Φ^i) , the Kähler form is

$$\omega = \begin{pmatrix} 0 & F_{ij} \\ -F_{ij} & 0 \end{pmatrix} .$$

We use in the following the fact that the Calabi-Yau metric cone is Ricci flat. From the classical formula

$$\rho = -i\partial\bar{\partial} \log \det(F_{ij}) ,$$

we get

$$\det(G_{ij}) = \exp \left(2\gamma_i \frac{\partial G}{\partial y^i} - c \right) , \quad (11)$$

with constants γ_i , and c . Using (9)-(10), we are able to express the coordinates x^i and the metric coefficients G_{ij}

$$\begin{aligned} x^i &= \frac{\partial G}{\partial y^i} = \frac{1}{2} \sum_A v_A^i \log l_A(y) + \frac{1}{2} B^i (1 + \log l_B(y)) \\ &\quad - \frac{1}{2} \sum_A v_A^i \log l_\infty(y) + \lambda_i, \\ G_{ij} &= \frac{1}{2} \sum_A \frac{v_A^i v_A^j}{l_A(y)} + \frac{1}{2} \frac{B_i B_j}{l_B(y)} - \frac{1}{2} \frac{\sum_A v_A^i \sum_A v_A^j}{l_\infty(y)}. \end{aligned}$$

Now, as the metric has to be smooth, from (11) it turns out that [15]

$$\gamma = (-1, 0, \dots, 0),$$

and

$$\det(F_{ij}) = \exp(2x^1 + c).$$

If Vol is the volume form on the metric cone, then the holomorphic volume form Ω satisfies

$$\text{Vol} = \frac{i^{n+1}}{2^{n+1}} (-1)^{n(n+1)/2} \Omega \wedge \bar{\Omega} = \frac{1}{(n+1)!} \omega^{n+1}.$$

Then, eventually ignoring the multiplicative constant, in complex coordinates Ω can be written as [13]

$$\begin{aligned} \Omega &= \exp(i\alpha) \det(F_{ij})^{1/2} dz^1 \wedge \dots \wedge dz^n \\ &= \exp(x^1 + i\alpha) dz^1 \wedge \dots \wedge dz^n. \end{aligned}$$

As Ω is parallel, it is also closed. Then we can fix the phase α to be Φ^1 , and we obtain the following simple formula for the holomorphic volume form, which motivates the interest for complex (and consequently, symplectic) coordinates

$$\Omega = \exp(z^1) dz^1 \wedge \dots \wedge dz^n. \quad (12)$$

Employing the above relation, in the next sections we show that it is possible to extract the special Killing forms on manifolds of Sasaki-Einstein type.

3 Symplectic and complex coordinates on conifold $C(T^{1,1})$

Throughout this section we introduce complex coordinates on $C(T^{1,1})$ using the classical procedure exposed above.

We start out by considering the global defined contact 1-form

$$\eta = \frac{1}{3} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2). \quad (13)$$

This form allows us to construct on $C(T^{1,1})$ the symplectic form (see e.g. [11])

$$\begin{aligned}\omega = & -\frac{r^2}{6}(\sin\theta_1 d\theta_1 \wedge d\phi_1 + \sin\theta_2 d\theta_2 \wedge d\phi_2) \\ & + \frac{1}{3}rdr \wedge (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2).\end{aligned}$$

Furthermore, if we employ the basis [11] for an effectively acting \mathbb{T}^3 action

$$\begin{aligned}e_1 &= \frac{\partial}{\partial\phi_1} + \frac{1}{2}\frac{\partial}{\partial\nu}, \\ e_2 &= \frac{\partial}{\partial\phi_2} + \frac{1}{2}\frac{\partial}{\partial\nu}, \\ e_3 &= \frac{\partial}{\partial\nu},\end{aligned}$$

where $2\nu = \psi$, then, considering action coordinates associated with this basis, we get the momentum map using (7) (see also [11])

$$\mu = \left(\frac{1}{6}r^2(\cos\theta_1 + 1), \frac{1}{6}r^2(\cos\theta_2 + 1), \frac{1}{3}r^2 \right).$$

The Reeb vector field B has the form

$$B = 3\frac{\partial}{\partial\psi} = \frac{3}{2}\frac{\partial}{\partial\nu}, \quad (14)$$

and is easy to see that $\eta(B) = 1$.

Now let us consider the “inward pointing” primitive normal vectors to the cone

$$v'_1 = [-1, 0, 1], v'_2 = [0, -1, 1], v'_3 = [1, 0, 0], v'_4 = [0, 1, 0].$$

We apply a $SL(3; \mathbb{Z})$ transformation T (see also [11])

$$T = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

to bring the vectors $v_i = Tv'_i$ in the form (8) [15]

$$v_1 = [1, 1, 1], v_2 = [1, 0, 1], v_3 = [1, 0, 0], v_4 = [1, 1, 0]. \quad (15)$$

According to the above transformation we obtain the new basis

$$\begin{aligned}e'_1 &= \frac{\partial}{\partial\phi_1} + \frac{1}{2}\frac{\partial}{\partial\nu}, \\ e'_2 &= \frac{\partial}{\partial\phi_2} - \frac{\partial}{\partial\phi_1}, \\ e'_3 &= -\frac{\partial}{\partial\phi_1} - \frac{\partial}{\partial\phi_2},\end{aligned} \quad (16)$$

where $e'_i := e_i T^{-1}$.

We consider the new angle coordinates

$$\begin{aligned}\Phi^1 &:= 2\nu = \psi, \\ \Phi^2 &:= -\frac{1}{2}\phi_1 + \frac{1}{2}\phi_2 + \nu = -\frac{1}{2}\phi_1 + \frac{1}{2}\phi_2 + \frac{1}{2}\psi, \\ \Phi^3 &:= -\frac{1}{2}\phi_1 - \frac{1}{2}\phi_2 + \nu = -\frac{1}{2}\phi_1 - \frac{1}{2}\phi_2 + \frac{1}{2}\psi.\end{aligned}$$

and it is easy to check that

$$e'_i = \frac{\partial}{\partial \Phi^i},$$

In this new basis, applying (7), the momentum map becomes:

$$\mu' = y = \left(\frac{1}{6}r^2(\cos \theta_1 + 1), \frac{1}{6}r^2(\cos \theta_2 - \cos \theta_1), -\frac{1}{6}r^2(\cos \theta_1 + \cos \theta_2) \right). \quad (17)$$

This way we end up with the symplectic action-angle coordinates (y^i, Φ^i) , for $1 \leq i \leq 3$.

Now, in order to introduce the complex coordinates on conifold we need the symplectic potential G . In the particular case of the conifold $T^{1,1}$, the sum $G^{can} + G^B$ is already a solution of the Monge-Ampère equation. Consequently the function h is not needed anymore, and the equations (9)-(10) simplify as (see e.g. [15])

$$G = G^{can} + G^B, \quad (18)$$

where

$$G^{can} = \sum_{A=1}^4 \frac{1}{2} \langle v_A, y \rangle \log \langle v_A, y \rangle,$$

and

$$G^B = \frac{1}{2} \langle B, y \rangle \log \langle B, y \rangle - \frac{1}{2} \langle B^{can}, y \rangle \log \langle B^{can}, y \rangle.$$

In the above relation the vectors v_A are just (15) and

$$B^{can} = \sum_1^4 v_A = (4, 2, 2).$$

Concerning the Reeb vector field on $T^{1,1}$ (14) written in the new basis $\{e'_i\}$ (16), it has the components

$$B = (3, 3/2, 3/2),$$

consistent with the determination from toric data using Z -minimization [9] or a -maximization [13].

We construct complex coordinates using (18) and the Legendre transform

$$x^i = \frac{\partial G}{\partial y^i} = \frac{1}{2} \sum_1^4 v_A^i \log \langle v_A, y \rangle + \frac{1}{2} B^i \log \langle B, y \rangle + \frac{1}{2} B^i - \frac{1}{2} (B^{can})^i \log \langle B^{can}, y \rangle .$$

But it is easy to see that

$$\begin{aligned} \sum_1^4 v_A^1 \log \langle v_A, y \rangle &= 8 \log r + 2 \log \sin \theta_1 + 2 \log \sin \theta_2 \\ &\quad - 4 \log 2 - 4 \log 3, \\ \sum_1^4 v_A^2 \log \langle v_A, y \rangle &= 4 \log r + \log(1 - \cos \theta_1) + \log(1 + \cos \theta_2) \\ &\quad - 2 \log 2 - 2 \log 3, \\ \sum_1^4 v_A^3 \log \langle v_A, y \rangle &= 4 \log r + \log(1 - \cos \theta_1) + \log(1 - \cos \theta_2) \\ &\quad - 2 \log 2 - 2 \log 3, \\ \log \langle B, y \rangle &= 2 \log r - \log 2, \\ \log \langle B^{can}, y \rangle &= 2 \log r + \log 2 - \log 3. \end{aligned} \tag{19}$$

Using now (19) and basic trigonometric formulas, we derive

$$\begin{aligned} x^1 &= 3 \log r + \log \sin \theta_1 + \log \sin \theta_2 + \frac{3}{2} - \frac{11}{2} \log 2, \\ x^2 &= \frac{3}{2} \log r + \log \sin \frac{\theta_1}{2} + \log \cos \frac{\theta_2}{2} + \frac{3}{4} - \frac{11}{4} \log 2, \\ x^3 &= \frac{3}{2} \log r + \log \sin \frac{\theta_1}{2} + \log \sin \frac{\theta_2}{2} - \frac{11}{4} \log 2. \end{aligned}$$

In the sequel, for the sake of simplicity we will ignore the additive constants. Therefore, in accordance with [13] we can introduce on conifold $C(T^{1,1})$ the following patch of complex coordinates

$$\begin{aligned} z^1 &= 3 \log r + \log \sin \theta_1 + \log \sin \theta_2 + i\psi, \\ z^2 &= \frac{3}{2} \log r + \log \sin \frac{\theta_1}{2} + \log \cos \frac{\theta_2}{2} \\ &\quad + \frac{i}{2}(\psi + \phi_1 + \phi_2), \\ z^3 &= \frac{3}{2} \log r + \log \sin \frac{\theta_1}{2} + \log \sin \frac{\theta_2}{2} \\ &\quad + \frac{i}{2}(\psi - \phi_1 - \phi_2). \end{aligned} \tag{20}$$

Now, regarding (12), we see that the above coordinates are precisely the necessary ingredient in order to extract the special Killing forms on our homogeneous Sasaki-Einstein manifold.

4 Special Killing forms on $T^{1,1}$

Applying the above general results, in this section we obtain the complete set of special Killing forms on the manifold $T^{1,1}$.

First of all, we have to calculate the holomorphic volume form Ω . Starting out with (20), we obtain

$$\begin{aligned}\exp(z^1) &= r^3 \sin \theta_1 \sin \theta_2 \exp i\psi, \\ dz^1 &= \frac{3}{r} dr + T_1, \\ dz^2 &= \frac{3}{2r} dr + T_2, \\ dz^3 &= \frac{3}{2r} dr + T_3.\end{aligned}$$

where

$$\begin{aligned}T_1 &:= \cot \theta_1 d\theta_1 + \cot \theta_2 d\theta_2 + i d\psi, \\ T_2 &:= \frac{1}{2} \cot \frac{\theta_1}{2} d\theta_1 - \frac{1}{2} \tan \frac{\theta_2}{2} d\theta_2 + \frac{i}{2} (d\psi - d\phi_1 + d\phi_2), \\ T_3 &:= \frac{1}{2} \cot \frac{\theta_1}{2} d\theta_1 + \frac{1}{2} \cot \frac{\theta_2}{2} d\theta_2 + \frac{i}{2} (d\psi - d\phi_1 - d\phi_2).\end{aligned}\tag{21}$$

Now we calculate the holomorphic volume form (see e.g. [13])

$$\begin{aligned}\Omega &= \exp(z^1) dz^1 \wedge dz^2 \wedge dz^3 \\ &= \exp(z^1) \left(\frac{3}{r} dr + T_1 \right) \wedge \left(\frac{3}{2r} dr + T_2 \right) \wedge \left(\frac{3}{2r} dr + T_3 \right).\end{aligned}$$

In our particular framework the equation (4) becomes

$$\Omega = r^2 dr \wedge \Psi + \frac{r^3}{3} d\Psi.$$

In order to extract Ψ we have to keep the trace of the differential form dr in the above equation. We clearly get

$$\Psi = 3 \sin \theta_1 \sin \theta_2 \exp(i\psi) \left(\frac{1}{4} T_2 \wedge T_3 - \frac{1}{2} T_1 \wedge T_3 + \frac{1}{2} T_1 \wedge T_2 \right).\tag{22}$$

We calculate the above wedge products using (21). For the first wedge product in (22), after calculations we get

$$\begin{aligned}
T_2 \wedge T_3 &= \frac{1}{2} \cot \frac{\theta_1}{2} \frac{1}{\sin \theta_2} d\theta_1 \wedge d\theta_2 - \frac{i}{2} \cot \frac{\theta_1}{2} d\theta_1 \wedge d\phi_2 \\
&\quad - \frac{i}{2} \frac{1}{\sin \theta_2} d\theta_2 \wedge d\psi + \frac{i}{2} \frac{1}{\sin \theta_2} d\theta_2 \wedge d\phi_1 - \frac{i}{2} \cot \theta_2 d\theta_2 \wedge d\phi_2 \\
&\quad - \frac{1}{2} d\phi_1 \wedge d\phi_2 + \frac{1}{2} d\psi \wedge d\phi_2.
\end{aligned} \tag{23}$$

For the second product we obtain

$$\begin{aligned}
T_1 \wedge T_3 &= \frac{1}{2} (\cot \theta_1 \cot \frac{\theta_2}{2} - \cot \theta_2 \cot \frac{\theta_1}{2}) d\theta_1 \wedge d\theta_2 - \frac{i}{2} \frac{1}{\sin \theta_1} d\theta_1 \wedge d\psi \\
&\quad - \frac{i}{2} \cot \theta_1 d\theta_1 \wedge d\phi_1 - \frac{i}{2} \cot \theta_1 d\theta_1 \wedge d\phi_2 - \frac{i}{2} \frac{1}{\sin \theta_2} d\theta_2 \wedge d\psi \\
&\quad - \frac{i}{2} \cot \theta_2 d\theta_2 \wedge d\phi_1 - \frac{i}{2} \cot \theta_2 d\theta_2 \wedge d\phi_2 + \frac{1}{2} d\psi \wedge d\phi_1 + \frac{1}{2} d\psi \wedge d\phi_2.
\end{aligned} \tag{24}$$

Finally, for the last product we get

$$\begin{aligned}
T_1 \wedge T_2 &= -\frac{1}{2} (\cot \theta_1 \tan \frac{\theta_2}{2} - \cot \theta_2 \cot \frac{\theta_1}{2}) d\theta_1 \wedge d\theta_2 - \frac{i}{2} \frac{1}{\sin \theta_1} d\theta_1 \wedge d\psi \\
&\quad - \frac{i}{2} \cot \theta_1 d\theta_1 \wedge d\phi_1 + \frac{i}{2} \cot \theta_1 d\theta_1 \wedge d\phi_2 + \frac{i}{2} \frac{1}{\sin \theta_2} d\theta_2 \wedge d\psi \\
&\quad - \frac{i}{2} \cot \theta_2 d\theta_2 \wedge d\phi_1 + \frac{i}{2} \cot \theta_2 d\theta_2 \wedge d\phi_2 + \frac{1}{2} d\psi \wedge d\phi_1 - \frac{1}{2} d\psi \wedge d\phi_2.
\end{aligned} \tag{25}$$

Now we plug (23)-(25) in (22) and we end up with the simple formula bellow for the special complex Killing form (in what follows we ignore the multiplicative constants)

$$\begin{aligned}
\Psi &= \exp(i\psi) [2d\theta_1 \wedge d\theta_2 - 2i \sin \theta_2 d\theta_1 \wedge d\theta_2 \\
&\quad + 2i \sin \theta_1 d\theta_2 \wedge d\phi_1 - 2 \sin \theta_1 \sin \theta_2 d\phi_1 \wedge d\phi_2].
\end{aligned}$$

From here, we can easily derive the real special Killing forms computing the real and imaginary part of Ψ :

$$\begin{aligned}
\Re \Psi &= \cos \psi d\theta_1 \wedge d\theta_2 + \sin \theta_2 \sin \psi d\theta_1 \wedge d\phi_2 \\
&\quad - \sin \theta_1 \sin \psi d\theta_2 \wedge d\phi_1 - \sin \theta_1 \sin \theta_2 \cos \psi d\phi_1 \wedge d\phi_2, \\
\Im \Psi &= \sin \psi d\theta_1 \wedge d\theta_2 - \sin \theta_2 \cos \psi d\theta_1 \wedge d\phi_2 \\
&\quad + \sin \theta_1 \cos \psi d\theta_2 \wedge d\phi_1 - \sin \theta_1 \sin \theta_2 \sin \psi d\phi_1 \wedge d\phi_2.
\end{aligned}$$

Finally, we calculate the Killing forms Φ_1 and Φ_2 (6), Ψ_1 and Ψ_2 (5) using the contact 1-form η (13). We obtain

$$\Phi_1 = d\eta = -\frac{1}{3} (\sin \theta_1 d\theta_1 \wedge d\phi_1 + \sin \theta_2 d\theta_2 \wedge d\phi_2),$$

$$\Phi_2 = (d\eta)^2 = -\frac{2}{9} \sin \theta_1 \sin \theta_2 d\theta_1 \wedge d\theta_2 \wedge d\phi_1 \wedge d\phi_2 ,$$

and respectively

$$\begin{aligned} \Psi_1 = \eta \wedge d\eta &= \frac{1}{9} (\sin \theta_1 d\psi \wedge d\theta_1 \wedge d\phi_1 + \sin \theta_2 d\psi \wedge d\theta_2 \wedge d\phi_2 \\ &\quad - \cos \theta_1 \sin \theta_2 d\theta_2 \wedge d\phi_1 \wedge d\phi_2 + \cos \theta_2 \sin \theta_1 d\theta_1 \wedge d\phi_1 \wedge d\phi_2) , \\ \Psi_2 = \eta \wedge (d\eta)^2 &= -\frac{2}{27} \sin \theta_1 \sin \theta_2 d\psi \wedge d\theta_1 \wedge d\theta_2 \wedge d\phi_1 \wedge d\phi_2 . \end{aligned}$$

5 Conclusions

The significance of the present work relies on unquestionable relevance of conformal Killing-Yano tensors in both mathematics and physics. It is known that it is a very difficult problem to find solutions of the conformal Killing-Yano equations on arbitrary Riemannian manifolds, but fortunately, in the case of spaces endowed with remarkable geometrical structures the explicit construction of the Killing forms is permitted.

In this paper we have obtained the complete set of special Killing forms on the five-dimensional Sasaki-Einstein space $T^{1,1}$ with an approach based on the interplay between symplectic and complex coordinates on Kähler toric manifolds. On the other hand, concerning the potential of the present paper, as a lot of non-trivial examples of toric Sasaki-Einstein manifolds occurs in the recent literature (see, e.g., [6, 12, 19]), it is both natural and useful to extend the present work to other spaces of interest. In fact, using toric geometry many examples of Sasaki-Einstein manifolds can be constructed, and these spaces are a good testing ground for the predictions of the AdS/CFT correspondence [14].

Acknowledgments

MV was supported by CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0137. The work of GEV was supported by CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0118.

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